Oxford Mathematics Alphabet

University of Oxford
Mathematical Institute

F *is for* Fourier transform

The Fourier transform is that rarest of things: a mathematical method from over 200 years ago which not only remains an active area of research in its own right, but is also an invaluable tool in nearly every branch of mathematics.

School students learn that one can express points in three-dimensional space $\mathbb{R}^3$ by three standard co-ordinates. The first-year undergraduate then learns that there are different sets of coordinates for the same point, depending on the *basis* for $\mathbb{R}^3$ which is chosen, and that a different basis could be better suited to the particular problem at hand. When the vector space is not $\mathbb{R}^3$ but rather a space of *functions* $f : G \longrightarrow \mathbb{C}$ for an abelian group G - the reals $\mathbb{R}$, the integers $\mathbb{Z}$, the unit circle $\mathbb{R}/\mathbb{Z}$, or the finite cyclic group $\mathbb{Z}/N\mathbb{Z}$, say - a particularly useful 'basis' is supplied by so-called *characters* on G. In the case of $\mathbb{R}$, these are continuous functions of the form $x \mapsto e^{2\pi ikx}$, for some fixed $k \in \mathbb{R}$. The 'coordinate' of a function f with respect to a phase $e^{2\pi ikx}$ is recorded by its Fourier transform $\widehat{f} : \mathbb{R} \longrightarrow \mathbb{C}$, defined as

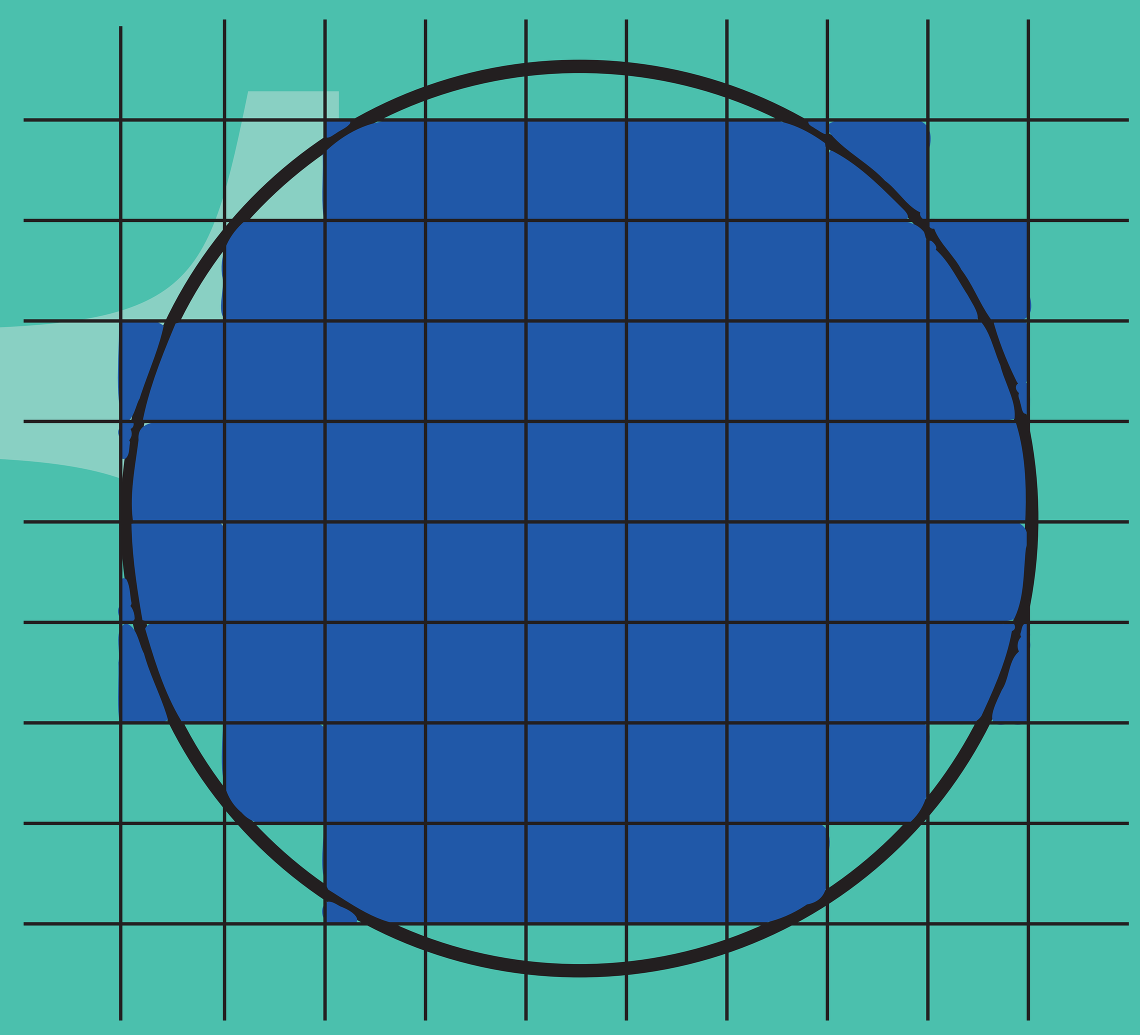
$$\widehat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ikx}\,dx.$$

Critically, this transformation can be inverted.

The applications to number theory are particularly surprising. The 'Gauss circle problem' asks us to find the number of integer lattice points inside a circle: for a fixed large r, we want to determine the number of integers m, n such that $m^2 + n^2 \leq r^2$. A fundamental property of the Fourier transform called *Poisson summation*, which states that

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{k\in\mathbb{Z}} \widehat{f}(k),$$

refines the error term, from roughly r (proved by analysing the diagram below) to roughly $r^{\frac{2}{3}}$.



You can even take the Fourier transform of the prime numbers. Using this method, the Russian mathematician Ivan Vinogradov proved that every (sufficiently large) odd number is the sum of three primes.

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For more about the Fourier transform visit www.maths.ox.ac.uk/r/alphabet